AEROSPACE VEHICLE STRUCTURES-1 (R17A2105)

Digital Notes

II B.TECH II SEM

(2018-2019)

Prepared by

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Department of Aeronautical Engineering

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

(Autonomous Institution- UGC, Govt. of India)

Affliated to JNTU, Hyderabad, Approved by AICTE-Accredited by NBA and NAAC-'A' Grade –ISO 9001:2015 Certified)
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MRCET VISION

To become a model institution in the fields of Engineering, Technology and Management.

To have a perfect synchronization of the ideologies of MRCET with challenging demands of International Pioneering Organizations.

MRCET MISSION

To establish a pedestal for the integral innovation, team spirit, originality and competence in the students, expose them to face the global challenges and become pioneers of Indian vision of modern society.

MRCET QUALITY POLICY.

To pursue continual improvement of teaching learning process of Undergraduate and Post Graduate programs in Engineering & Management vigorously.

To provide state of art infrastructure and expertise to impart the quality education.

PROGRAM OUTCOMES
(PO’s)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design / development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multi disciplinary environments.
12. **Life- long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

DEPARTMENT OF AERONAUTICAL ENGINEERING

VISION

Department of Aeronautical Engineering aims to be indispensable source in Aeronautical Engineering which has a zeal to provide the value driven platform for the students to acquire knowledge and empower themselves to shoulder higher responsibility in building a strong nation.

MISSION

The primary mission of the department is to promote engineering education and research. To strive consistently to provide quality education, keeping in pace with time and technology. Department passions to integrate the intellectual, spiritual, ethical and social development of the students for shaping them into dynamic engineers

QUALITY POLICY STATEMENT

Impart up-to-date knowledge to the students in Aeronautical area to make them quality engineers. Make the students experience the applications on quality equipment and tools. Provide systems, resources and training opportunities to achieve continuous improvement. Maintain global standards in education, training and services.

PROGRAM EDUCATIONAL OBJECTIVES – Aeronautical Engineering

1. **PEO1 (PROFESSIONALISM & CITIZENSHIP):** To create and sustain a community of learning in which students acquire knowledge and learn to apply it professionally with due consideration for ethical, ecological and economic issues.
2. **PEO2 (TECHNICAL ACCOMPLISHMENTS):** To provide knowledge based services to satisfy the needs of society and the industry by providing hands on experience in various technologies in core field.
3. **PEO3 (INVENTION, INNOVATION AND CREATIVITY):** To make the students to design, experiment, analyze, and interpret in the core field with the help of other multi disciplinary concepts wherever applicable.
4. **PEO4 (PROFESSIONAL DEVELOPMENT):** To educate the students to disseminate research findings with good soft skills and become a successful entrepreneur.
5. **PEO5 (HUMAN RESOURCE DEVELOPMENT):** To graduate the students in building national capabilities in technology, education and research

PROGRAM SPECIFIC OUTCOMES – Aeronautical Engineering

1. To mould students to become a professional with all necessary skills, personality and sound knowledge in basic and advance technological areas.
2. To promote understanding of concepts and develop ability in design manufacture and maintenance of aircraft, aerospace vehicles and associated equipment and develop application capability of the concepts sciences to engineering design and processes.
3. Understanding the current scenario in the field of aeronautics and acquire ability to apply knowledge of engineering, science and mathematics to design and conduct experiments in the field of Aeronautical Engineering.
4. To develop leadership skills in our students necessary to shape the social, intellectual, business and technical worlds.

II B.TECH II SEMESTER – AERONAUTICAL ENGINEERING

AEROSPACE VEHICLE STRUCTURES-1

UNIT-1

Introduction to Aircraft Structures

1. Structural components of aircraft

For an aircraft to fly, aerodynamics and propulsion is most important. But if we go back to a century back it is all the structure that played an important role. Aircraft structures are basically categorized what we see externally which are sometimes referred to as the primary aircraft components such as aircraft wing, fuselage, landing gear, stabilizers and control surfaces. We shall discuss different types of constructions in detail about these components along with their role in influencing the aircraft structural stability.

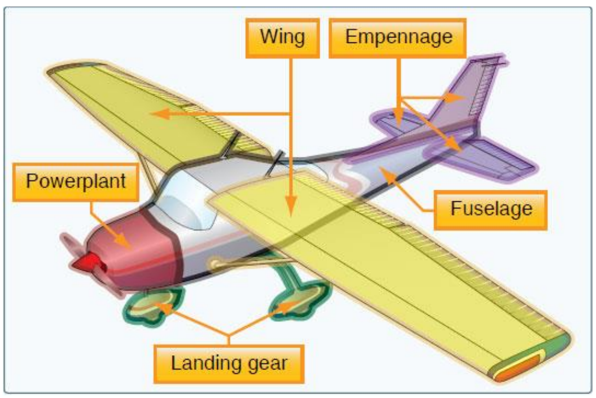


Figure 1: structural components of aircraft

Fuselage

Aircraft fuselages consist, of thin sheets of material stiffened by large numbers of longitudinal stringers together with transverse frames. Generally, they carry bending moments, shear forces and torsional loads which induce axial stresses in the stringers and skin together with shear stresses in the skin; the resistance of the stringers to shear forces is generally ignored. Also, the distance between adjacent stringers is usually small so that the variation in shear flow in the connecting panel will be small. It is therefore reasonable to assume that the shear flow is constant between adjacent stringers so that the analysis simplifies to the analysis of an idealized section in which the stringers/booms carry all the direct stresses while the skin is effective only in shear.

There are three most common type of fuselage construction as discussed below

- Truss of framework type

This consists of light gauge steel tubes which form a frame triangular shape to give the most rigid of geometric forms. Each tube carries a specific load, the magnitude of which depends on whether the aircraft is airborne or on the ground. This type of fuselage is commonly found on the first few generations of aircraft. They are strong, moderately easy to manufacture, but did not necessarily implement the concept of aerodynamic.

Figure 2: Truss type, monocoque and semi-monocoque structure

- Monocoque Construction

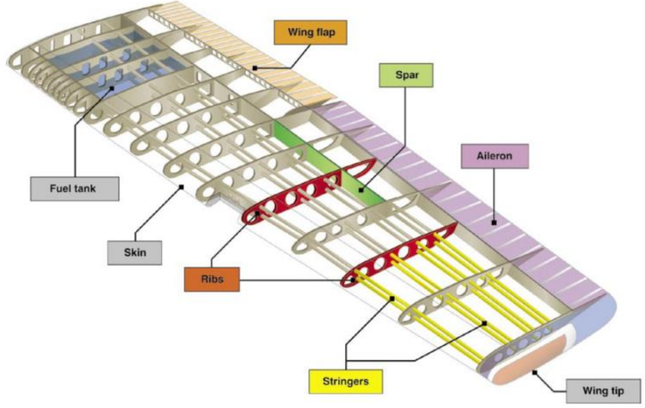
'Monocoque' is a French word meaning 'single shell'. All the loads are taken by a stressed skin with just light internal frames or formers to give the required shape. To be a true, Monocoque structure would have no apertures at all. Although it practically can carry more loads, the drawback of this type is that it may require maintenance more compared to the other designs, as the structure needs to be reinforced in order to maintain the structural integrity.

- Semi-Monocoque Construction

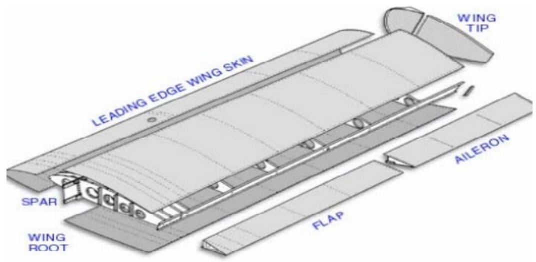
As aircraft became larger, the pure Monocoque was found not to be strong enough. Designers came with a new concept to make fuselage stronger; the Longerons run lengthwise along the fuselage joining the frames together (see picture below for more detail). The light alloy skin is attached to the frames and longerons by riveting or adhesive bonding. Doublers are required when cut-outs are made to provide access panels, doors or windows. Bulkheads isolate different sections of the aircraft, for instance the engine compartment from the passenger compartment. Bulkheads are of much stronger construction than frames or formers, as the loads upon them are so much greater. This concept is widely used both in military and also in the commercial industry. In military, this concept is believed to enable planes to gain more speed.

Wing

Wing sections consist of thin skins stiffened by combinations of stringers, spar webs, and caps and ribs. The resulting structure frequently comprises one, two or more cells, and is highly redundant. However, as in the case of fuselage sections, the large number of closely spaced stringers allows the assumption of a constant shear flow in the skin between adjacent stringers so that a wing section may be analyzed as though it were completely idealized as long as the direct stress carrying capacity of the skin is allowed for by additions to the existing stringer/boom areas.



Landing gear



2. Geodesic airframe

A geodesic (or geodetic) airframe is a type of construction for the airframes of aircraft developed by British aeronautical engineer Barnes Wallis in the 1930s. It makes use of a space frame formed from a spirally crossing basket-weave of load-bearing members. The principle is that two geodesic arcs can be drawn to intersect on a curving surface (the fuselage) in a manner that the torsional load on each cancels out that on the other.

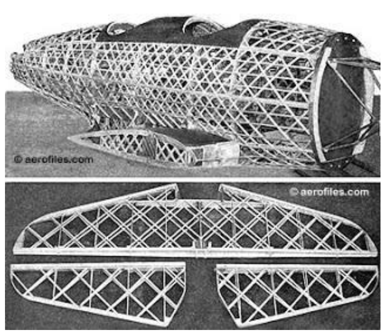


Figure 3: geodesic construction of airframe structures

UNIT-2

Deflection of Beams

DEFLECTION OF DETERMINATE BEAMS

Assumptions of elementary beam theory

The beam theory is used in the design and analysis of a wide range of structures, from buildings to bridges to the load bearing bones of the human body.

- Material of beam is homogenous and isotropic => constant E in all direction
- Young's modulus is constant in compression and tension => to simplify analysis
- Transverse sections which are plane before bending before bending remain plain after bending. => Eliminate effects of strains in other direction.
- Beam is initially straight and all longitudinal filaments bend in circular arcs => simplify calculations
- Radius of curvature is large compared with dimension of cross sections => simplify calculations
- Each layer of the beam is free to expand or contract => Otherwise they will generate additional internal stresses.

Deflection of beams

In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

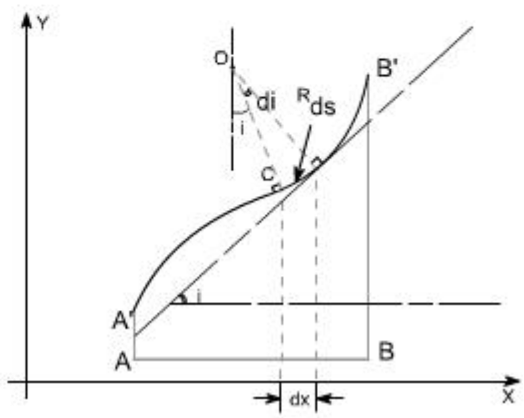
1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.

2. The curvature is always small.

3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

4. It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.

Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.
In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Further, it is assumed that the simple bending theory equation holds good.



$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points. To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection. Further, let us consider element ***ds*** of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be ***di***
But for the deflected shape of the beam the slope ***i*** at any point C is defined,

$$tan i = \frac{dy}{dx}$$

Further $ds = Rdi$, however ds=dx (usually for small curvature) resulting in $\frac{di}{dx} = \frac{1}{R}$, substituting the value of ***i,*** and arranging it with flexural equation above, we get

$$M = EI\frac{d^2y}{dx^2}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

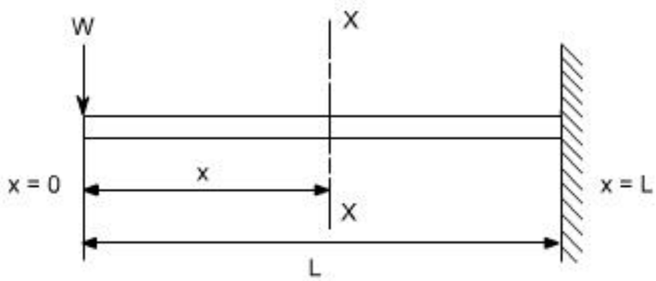
Relationship between shear force, bending moment and deflection

Differentiating the equation as derived

Deflection	EIy
Slope	$EI\frac{dy}{dx}$
Bending Moment	$EI\frac{d^2y}{dx^2}$
Shear force	$EI\frac{d^3y}{dx^3}$
Load distribution	$EI\frac{d^4y}{dx^4}$

Method of double integration: The primary advantage of the double- integration method is that it produces the equation for the deflection everywhere along the beams.

Cantilever Beam with Concentrated Load at the end:- A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force and the bending moment.

$S.F|_{x-x} = -W$

$B.M|_{x-x} = -W.x$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

$$i.e\ at\ x = L;\ y = 0$$

$$at\ x = L; \frac{dy}{dx} = 0$$

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

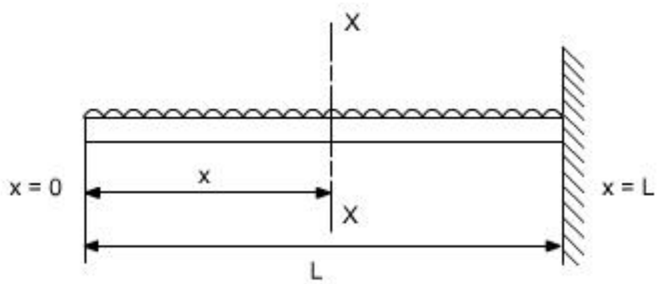
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(Slope)_{max}m = +\frac{WL^2}{2EI}}$$

A Cantilever with Uniformly distributed Loads:- In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length. The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int \frac{dy}{dx} = \int -\frac{wx^3}{6EI}dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

$$At\ x = L;\ y = 0$$

$$At\ x = L;\ dy/dx = 0$$

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

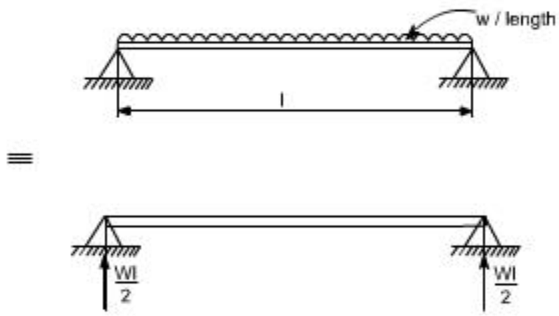
$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$$

So y_{maxm} will be at $x = 0$

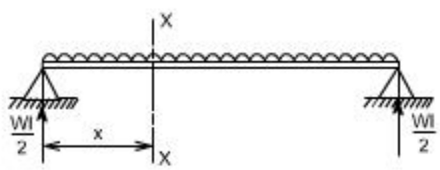
$$\boxed{y_{maxm} = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{maxm} = \frac{wL^3}{6EI}}$$

<u>Simply Supported beam with uniformly distributed Loads:-</u> In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x meter from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad ----(1)$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. $at\ x = 0;\ y = 0 :\ at\ x = l;\ y = 0$

Let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, this yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

Then $$y_{maxm} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{maxm} = -\frac{5wL^4}{384EI}}$$

Conclusions

(i) The value of the slope at the position where the deflection is maximum would be zero.

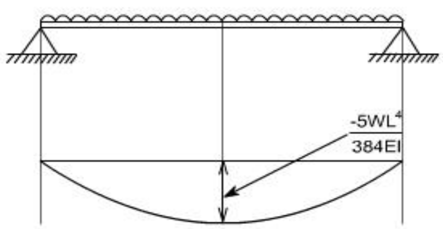
(ii) The value of maximum deflection would be at the center i.e. at x = L/2.

The final equation which is governs the deflection of the loaded beam in this case is

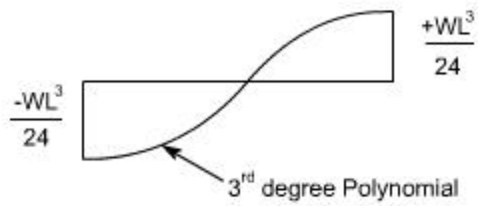
$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.

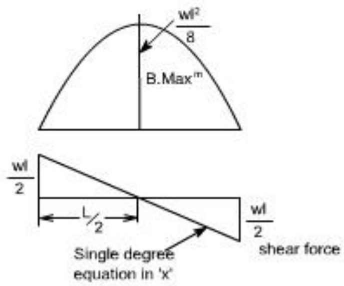
Deflection (y) $$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$



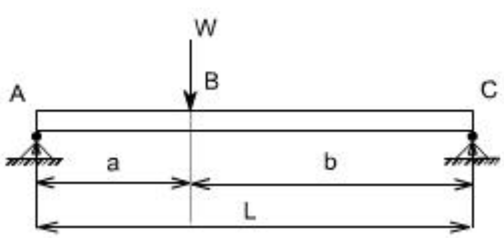
Slope (dy/dx) $EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$



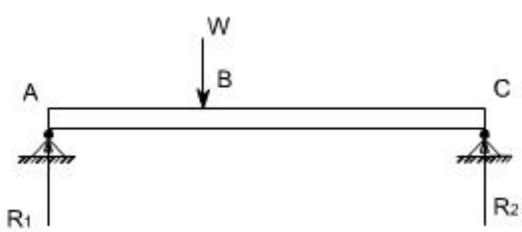
Bending Moment $\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$



Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting entire beam **:** The direct integration method may become more involved if the expression for entire beam is not valid for the at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$M|_{AB} = R_1.x \quad 0 \le x \le a$

B.M for the portion BC

$M|_{BC} = R_1.x - W(x - a) \quad a \le x \le l$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1 x$$

$$EI\frac{d^2y}{dx^2} = R_1 x - W(x - a)$$

These two equations can be integrated in the usual way to find 'y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at x = 0; y = 0 in the portion AB i.e. $0 \le x \le a$

(b) at x = l; y = 0 in the portion BC i.e. $a \le x \le l$

(c) at x = a; dy/dx, the slope is same for both portion

(d) at x = a; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting

$K_1 = K_2 = K$

Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{ ------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{ ------- (4)}$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{ ------(5)}$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{ ------(6)}$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a + b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y|_{\text{from equation 5}} = y|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{----} \mathbf{for\ 0 \le x \le a} \text{-----(7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{For } \mathbf{for\ a \le x \le l} \text{-----(8)}$$

so either of the equation (7) or (8) may be used to find the deflection at x = a

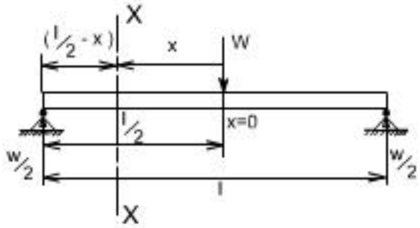
hence substituting x = a in either of the equation we get

$$Y|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if a = b = l/2

$$\boxed{Y_{max^m} = -\frac{WL^3}{48EI}}$$

<u>ALTERNATE METHOD:</u> There is also an alternative way to attempt this problem in a simpler way. Let us considering the origin at the point of application of the load,



$$S.F|_{xx} = \frac{W}{2}$$

$$B.M|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we have taken the origin at the centre)

Thus,

$$0 = \left[\frac{WL^3}{32} - \frac{WL^3}{96} + B\right]$$

$$B = -\frac{WL^3}{48}$$

Hence he equation which governs the deflection would be

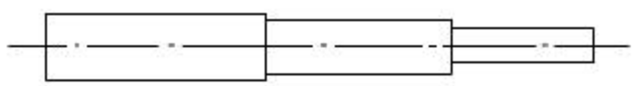
$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right]$$

Hence

$$Y_{max^m}\Big|_{at\,x=0} = -\frac{WL^3}{48EI} \quad \text{At the centre}$$

$$\left(\frac{dy}{dx}\right)_{max^m}\Big|_{at\,x=\pm\frac{L}{2}} = \pm\frac{WL^2}{16EI} \quad \text{At the ends}$$

Hence the integration method may be bit cumbersome in some of the case. Another limitation of the method would be that if the beam is of non uniform cross section,



i.e. it is having different cross-section then this method also fails.

So there are other methods by which we find the deflection like

1. Macaulay's method in which we can write the different equation for bending moment for different sections.

2. Area moment methods

3. Energy principle methods

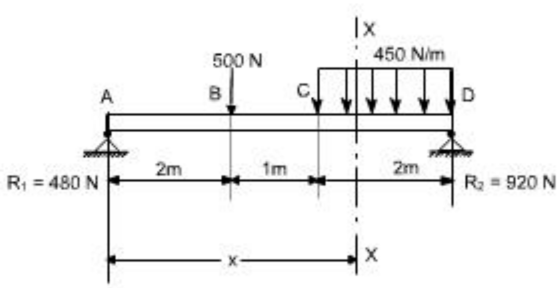
Macaulay's method

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that to integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

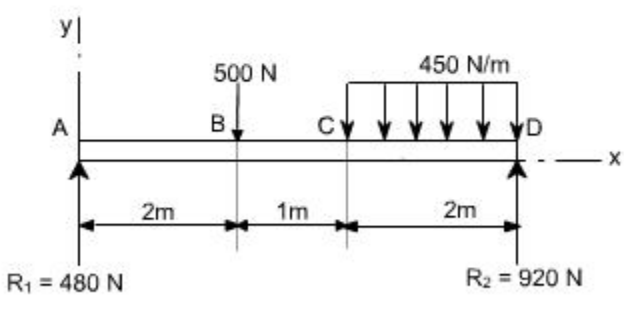
For example consider the beam shown in fig below:

Let us write the general moment equation using the definition $M = (\sum M)_L$, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows



$$M_{AB} = 480\,x\ \text{N.m}$$
$$M_{BC} = \left[480\,x - 500\,(x-2)\right]\text{N.m}$$
$$M_{CD} = \left[480\,x - 500\,(x-2) - \frac{450}{2}(x-3)^2\right]\text{N.m}$$

It may be observed that the equation for M_{CD} will also be valid for both M_{AB} and M_{BC} provided that the terms (x - 2) and $(x - 3)^2$ are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and $(x - 3)^2$ are nonexistent for values of x for which the terms in parentheses are negative.

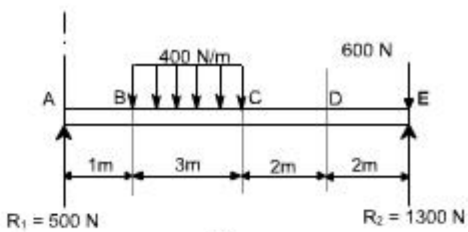


As an clear indication of these restrictions,one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

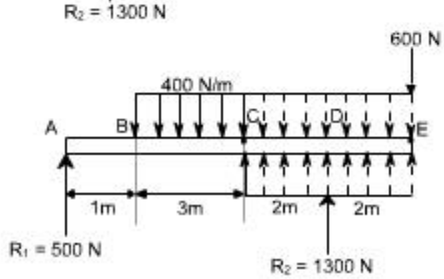
$$M = \left(480x - 500\left(x-2\right) - \frac{450}{2}\left(x-3\right)^2\right)N.m$$

Which is valid for the entire beam if we postulate that the terms between the pointed brackets do not exists for negative values; otherwise the term is to be treated like any ordinary expression.

As an example, consider the beam as shown in the fig below. Here the distributed load extends only over the segment BC. We can create continuity, however, by assuming that the distributed load extends beyond C and adding an equal upward-distributed load to cancel its effect beyond C, as shown in the adjacent fig below. The general moment equation, written for the last segment DE in the new nomenclature may be written as:



(a)



$$M = \left(500x - \frac{400}{2}\left(x-1\right)^2 + \frac{400}{2}\left(x-4\right)^2 + 1300\left(x-6\right)\right)N.m$$

It may be noted that in this equation effect of load 600 N won't appear since it is just at the last end of the beam so if we assume the exploratory just at section at just the point of application of 600 N than x = 0 or else we will here take the X - section beyond 600 N which is invalid.

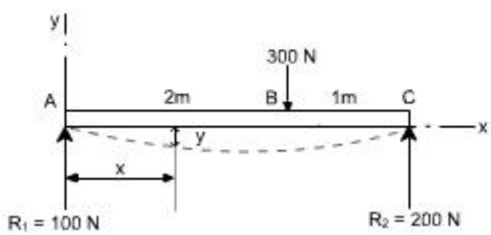
Procedure to solve the problems

(i). After writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

llustrative Examples :

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig. Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.



Solution : writing the general moment equation for the last portion BC of the loaded beam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right)N.m \qquad ....(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right)N.m^2 \qquad ....(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)N.m^3 \qquad ....(3)$$

To evaluate the two constants of integration, Let us apply the following boundary conditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these values in Eq. (3) we find $C_2 = 0$.keep in mind that $< x -2 >^3$ is to be neglected for negative values.

2. At the other support where x = 3m, the value of deflection y is also zero.

Substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50(3-2)^3 + 3.C_1\right) \text{or } C_1 = -133\text{N.m}^2$$

Having determined the constants of integration, let us make use of Eqs. (2) and (3) to rewrite the slope and deflection equations in the conventional form for the two portions.

segment AB ($0 \le x \le 2$m)

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right)\text{N.m}^2 \qquad ....(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right)\text{N.m}^3 \qquad ....(5)$$

segment BC (2m $\le x \le$ 3m)

$$EI\frac{dy}{dx} = \left(50x^2 - 150(x-2)^2 - 133x\right)\text{N.m}^2 \,.......(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50(x-2)^3 - 133x\right)\text{N.m}^3 \;.......(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

We obtain

$50\ x^2 – 133 = 0$ or $x = 1.63$ m (It may be kept in mind that if the solution of the equation does not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x = 1.63 m in Eq (5), which yields

$$EIy\big|_{max} m = -145\text{N.m}^3 \qquad ....(8)$$

The negative value obtained indicates that the deflection y is downward from the x axis.quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by d, the use of y may be reserved to indicate a directed value of deflection.

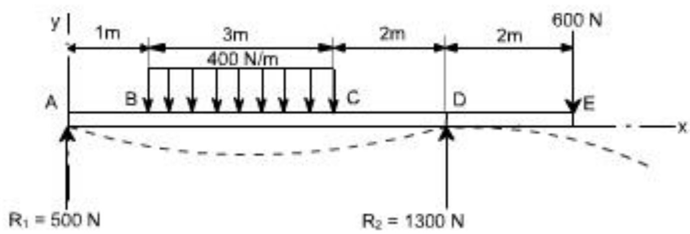
if E = 30 Gpa and $I = 1.9 \times 10^6\ mm^4 = 1.9 \times 10^{-6}\ m^4$, Eq. (h) becomes

$$y|_{max} m = (30 \times 10^9)(1.9 \times 10^{-6})$$

Then $= -2.54 mm$

Example 2:

It is required to determine the value of EIy at the position midway between the supports and at the overhanging end for the beam shown in figure below.



Solution:

Writing down the moment equation which is valid for the entire span of the beam and applying the differential equation of the elastic curve, and integrating it twice, we obtain

$$EI\frac{d^2y}{dx^2} = M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right) N.m$$

$$EI\frac{dy}{dx} = \left(250x^2 - \frac{200}{3}(x-1)^3 + \frac{200}{3}(x-4)^3 + 650(x-6)^2 + C_1\right) N.m$$

$$EIy = \left(\frac{250}{3}x^3 - \frac{50}{3}(x-1)^4 + \frac{50}{3}(x-4)^4 + \frac{650}{3}(x-6)^3 + C_1x + C_2\right) N.m^3$$

To determine the value of C_2, It may be noted that EIy = 0 at x = 0,which gives C_2 = 0.Note that the negative terms in the pointed brackets are to be ignored Next,let us use the condition that EIy = 0 at the right support where x = 6m.This gives

$$0 = \frac{250}{3}(6)^3 - \frac{50}{3}(5)^4 + \frac{50}{3}(2)^4 + 6C_1 \quad \text{or} \quad C_1 = -1308 N.m^2$$

Finally, to obtain the midspan deflection, let us substitute the value of x = 3m in the deflection equation for the segment BC obtained by ignoring negative values of the bracketed terms á x - 4 ñ^{4} and á x - 6 ñ^{3}. We obtain

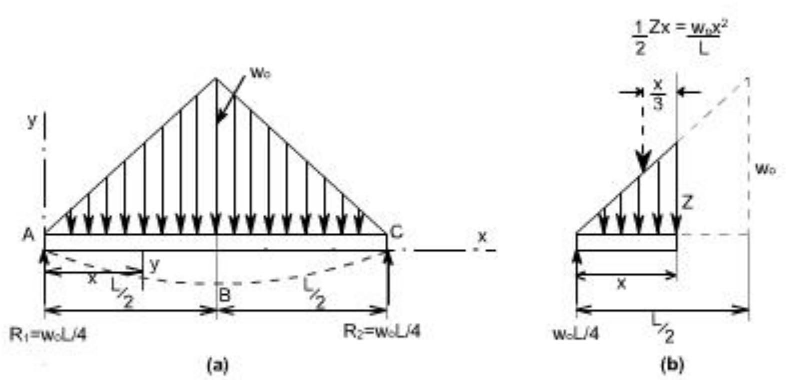
$$EIy = \frac{250}{3}(3)^3 - \frac{50}{3}(2)^4 - 1308(3) = -1941\ N.m^3$$

For the overhanging end where x=8 m,we have

$$EIy = \left(\frac{250}{3}(8)^3 - \frac{50}{3}(7)^4 + \frac{50}{3}(4)^4 + \frac{650}{3}(2)^3 - 1308(8)\right)$$
$$= -1814 N.m^3$$

Example 3:

A simply supported beam carries the triangularly distributed load as shown in figure. Determine the deflection equation and the value of the maximum deflection.



Solution:

Due to symmetry, the reactions is one half the total load of $1/2w_0L$, or $R_1 = R_2 = 1/4w_0L$.Due to the advantage of symmetry to the deflection curve from A to B is the mirror image of that from C to B. The condition of zero deflection at A and of zero slopes at B does not require the use of a general moment equation. Only the moment equation for segment AB is needed and this may be easily written with the aid of figure (b).

Taking into account the differential equation of the elastic curve for the segment AB and integrating twice, one can obtain

$$EI\frac{d^2y}{dx^2} = M_{AB} = \frac{w_0L}{4}x - \frac{w_0x^2}{L}\cdot\frac{x}{3} \qquad ......(1)$$

$$EI\frac{dy}{dx} = \frac{w_0Lx^2}{8} - \frac{w_0x^4}{12L} + C_1 \qquad ......(2)$$

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} + C_1x + C_2 ......(3)$$

In order to evaluate the constants of integration, let us apply the B.C's we note that at the support A, y = 0 at x = 0.Hence from equation (3), we get $C_2 = 0$. Also, because of symmetry, the slope dy/dx = 0 at midspan where x = L/2.Substituting these conditions in equation (2) we get

$$0 = \frac{w_0L}{8}\left(\frac{L}{2}\right)^2 - \frac{w_0}{12L}\left(\frac{L}{2}\right)^4 + C_1 C_1 = -\frac{5w_0L^3}{192}$$

Hence the deflection equation from A to B (and also from C to B because of symmetry) becomes

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} - \frac{5w_0L^3x}{192}$$

Whichreducesto

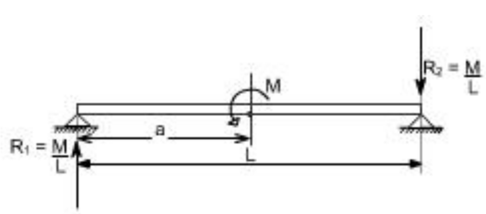
$$EIy = -\frac{w_0x}{960L}\left(25L^4 - 40L^2x^2 + 16x^4\right)$$

The maximum deflection at midspan,where x = L/2 is then found to be

$$\boxed{EIy = -\frac{w_0L^4}{120}}$$

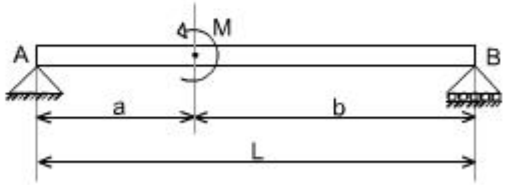
Example 4: couple acting

Consider a simply supported beam which is subjected to a couple M at a distance 'a' from the left end. It is required to determine using the Macauley's method.



To deal with couples, only thing to remember is that within the pointed brackets we have to take some quantity and this should be raised to the power zero. I.e. M á x - a ñ^0. We have taken the power 0 (zero) ' because ultimately the term M á x - a ñ^0 should have the moment units. Thus with integration the quantity á x - a ñ becomes either á x - a ñ^{1}or á x - a ñ^2

Or



Therefore, writing the general moment equation we get

$$M = R_1 x - M\langle x - a\rangle \text{ or } EI\frac{d^2y}{dx^2} = M$$

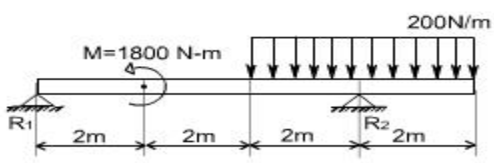
Integrating twice we get

$$EI\frac{dy}{dx} = R_1.\frac{x^2}{2} - M\langle x - a\rangle^1 + C_1$$

$$EI.y = R_1.\frac{x^3}{6} - \frac{M}{2}\langle x - a\rangle^2 + C_1 x + C_2$$

Example 5:

A simply supported beam is subjected to U.d.l in combination with couple M. It is required to determine the deflection.



This problem may be attempted in the same way. The general moment equation my be written as

$$M(x) = R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle\langle x-4\rangle}{2} + R_2\langle x-6\rangle$$

$$= R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle^2}{2} + R_2\langle x-6\rangle$$

Thus,

$$EI\frac{d^2y}{dx^2} = R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle^2}{2} + R_2\langle x-6\rangle$$

Integrate twice to get the deflection of the loaded beam.

DEFLECTION OF REDUNDANT STRUCTURES

A structure in which the laws of statics are not sufficient to determine all the unknown forces or moments is said to be statically indeterminate. Such structures are analyzed by writing the appropriate equations of static equilibrium and additional equations pertaining to the deformation and constraints known as compatibility condition.

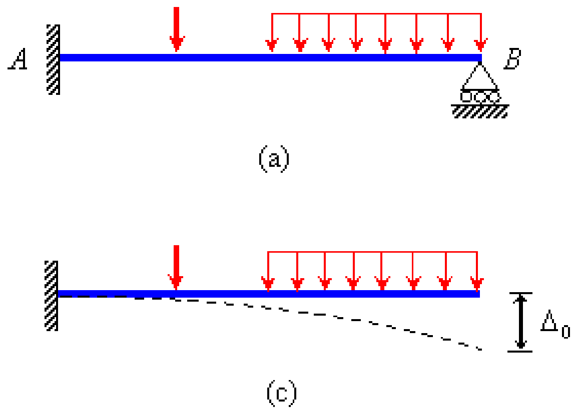
The statically indeterminate structures are frequently used for several advantages. They are relatively more economical in the requirement of material as the maximum bending moments in the structure are reduced. The statically indeterminate are more rigid leading to smaller deflections. The disadvantage of the indeterminate structure is that they are subjected to stresses when subjected to temperature changes and settlements of the support. The construction of indeterminate structure is more difficult if there are dimensional errors in the length of members or location of the supports.

This chapter deals with analysis of statically indeterminate structures using various force methods.

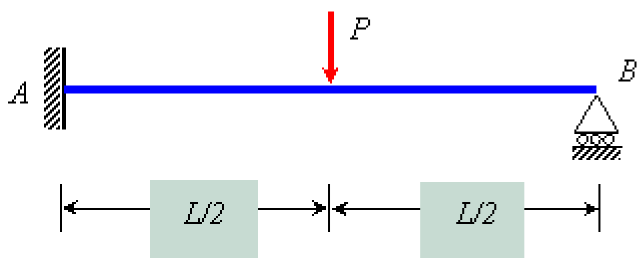
Analysis of Statically Indeterminate Beams

The moment area method and the conjugate beam method can be easily applied for the analysis of statically indeterminate beams using the principle of superposition. Depending upon the degree of indeterminacy of the beam, designate the excessive reactions as redundant and modify the support. The redundant reactions are then treated as unknown forces. The redundant reactions should be such that they produce the compatible deformation at the original support along with the applied loads. For example consider a propped cantilever beam as shown in

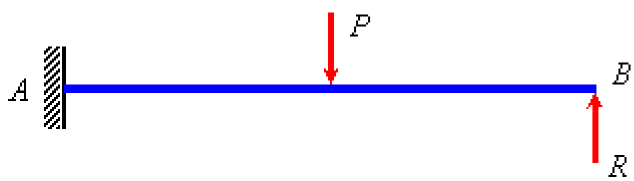
Figure 1(a). Let the reaction at B be R as shown in Figure 1(b) which can be obtained with the compability condition that the downward vertical deflection of B due to applied loading (i.e. Shown in Figure 1(c)) should be equal to the upward vertical deflection of B due to R (i.e. shown in Figure 1(d)).



Determine the support reactions of the propped cantilever beam as shown in Figure 2(a).



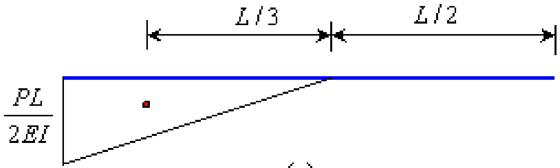
The static indeterminacy of the beam is = 3 – 2 = 1. Let reaction at *B* is *R* acting in the upward direction as shown in Figure 2(b). The condition available is that the $\Delta_B = 0$.



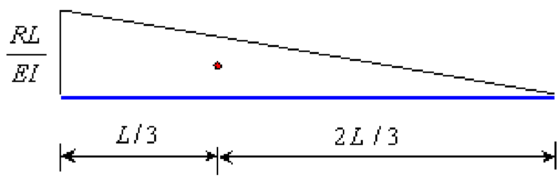
Moment area method

The bending moment diagrams divided by EI of the beam are shown due to P and R in Figures 2(c) and (d), respectively.

Since in the actual beam the deflection of the point B is zero which implies that the deviation of point B from the tangent at A is zero. Thus,



(c)



(d)

$$t_{BA} = 0$$

$$-\frac{1}{2}\times\frac{L}{2}\times\frac{PL}{2EI}\left(\frac{L}{2}+\frac{L}{3}\right)+\frac{1}{2}\times L\times\frac{RL}{EI}\left(\frac{2L}{3}\right)=0$$

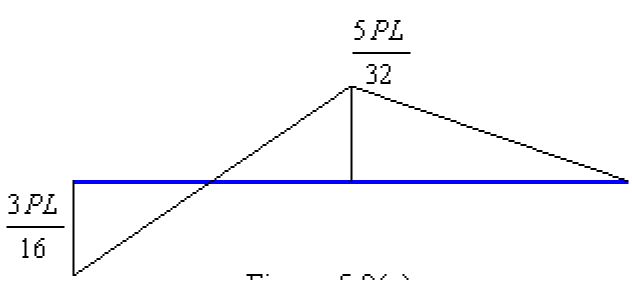
$$A_{m_1}=\frac{AE}{L}\left[(D_{m_3}-D_{m_1})C_x+(D_{m_4}-D_{m_2})C_y\right]$$

Taking moment about A , the moment at A is given by

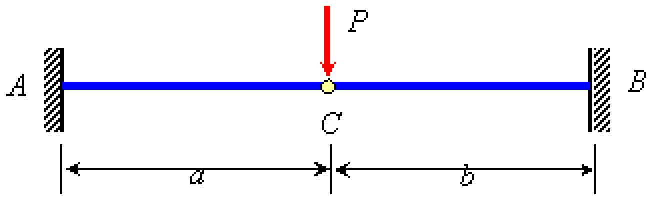
$$M_A = P\times\frac{L}{2}-\frac{5P}{16}\times L=\frac{3PL}{16}$$

The vertical reaction at A is

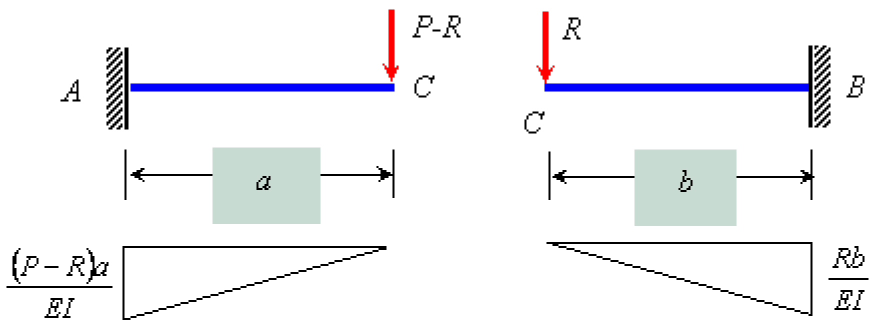
$$V_A = P-\frac{5P}{16}=\frac{11P}{16}$$



Determine the support reactions of the fixed beam with internal hinge as shown in Figure



The static indeterminacy of the beam is = 4-2-1 =1 Let the shear in the internal hinge be R. The free body diagrams of the two separated portions of the beam are shown in Figure 3(b) along with their M/EI diagrams. The unknown R can be obtained with the condition that the vertical deflection of the free ends of the two separated cantilever beams is identical.



Consider AC: The vertical displacement of C is given by

$$\Delta_C = t_{CA} = -\frac{1}{2} \times a \times \frac{(P-R)a}{EI} \times \frac{2a}{3}$$

Consider CB: The vertical displacement of C is given by

$$\Delta_C = t_{CB} = -\frac{1}{2} \times b \times \frac{Rb}{EI} \times \frac{2b}{3}$$

Solving for R will give

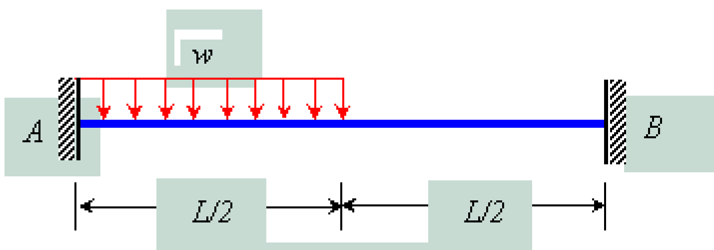
$$R = \frac{Pa^3}{(a^3 + b^3)}$$

The reactions at the supports are given by

$$V_A = \frac{Pb^1}{(a^1 + b^1)} \quad \text{and} \quad M_A = \frac{Pb^3 a}{(a^3 + b^3)}$$

$$V_B = \frac{Pa^1}{(a^1 + b^1)} \quad \text{and} \quad M_B = \frac{Pa^3 b}{(a^3 + b^3)}$$

Determine the support reactions of the fixed beam as shown in Figure 5(a). The beam carries a uniformly distributed load, w over the left half span.



The static indeterminacy of the beam is = 4-2 =2. Let the reactions at B be the unknown as shown in Figure

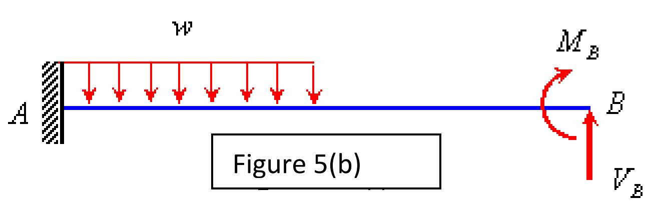


Figure 5(b)

a. Moment Area Method

The free body diagram of the beam is shown below along with their M/EI diagrams. The unknowns V_A and V_B can be obtained with the condition that the vertical deflection and slope at B are zero.

Since the change of slope between points A and B is zero (due to fixed supports at A and B), therefore, according to the first moment area theorem,

$$\Delta\theta_{BA} = 0$$

$$\Delta\theta_{BA} = \left(-\frac{1}{3}\times\frac{L}{2}\times\frac{wL^2}{8EI}\right) + \left(\frac{1}{2}\times L\times\frac{V_B L}{EI}\right) - \left(\frac{M_B}{EI}\times L\right) = 0$$

$$\frac{V_B L}{2} - M_B = \frac{wL^2}{48} \quad , \quad t_{AB} = 0$$

$$t_{AB} = \left(-\frac{1}{3} \times \frac{L}{2} \times \frac{wL^2}{8EI}\right) \times \frac{L}{8} + \left(\frac{1}{2} \times L \times \frac{V_B L}{EI}\right) \times \frac{L}{3} - \left(\frac{M_B}{EI} \times L\right) \times \frac{L}{2} = 0$$

$$\frac{V_B L}{6} - \frac{M_B}{2} = \frac{wL^2}{384}$$

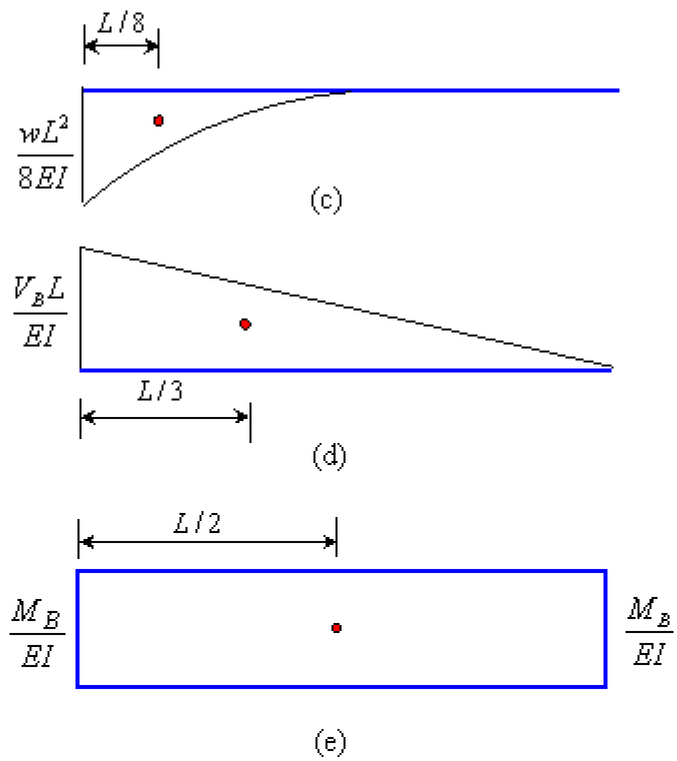


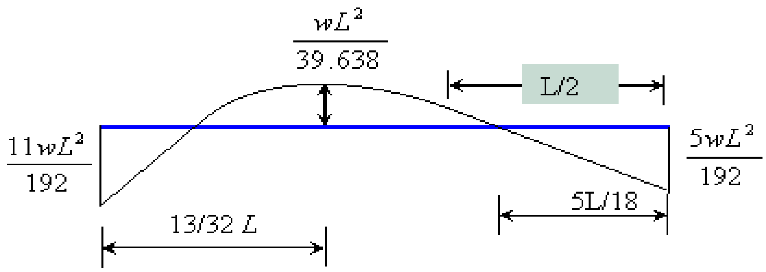
Figure 5.5 *M/EI* diagram due to (c) applied external load, (d) V_B and (e) due to M_B

Solving equations (i) and (ii)

$$V_B = \frac{3}{32} wL \text{ and } M_A = \frac{11}{192} wL^2$$

$$V_A = \frac{13}{32} wL \text{ and } M_B = \frac{5}{192} wL^2$$

The bending moment diagram of the beam is shown in Figure 5(f)



The end B of a uniform fixed beam sinks by an amount D. Determine the end reactions using moment area method.

The degree of indeterminacy is 2. Let end reactions due to settlement at B be V_B and M_B as shown in Figure 6(b). The M/EI diagram of the beam is shown in Figure 6(c).

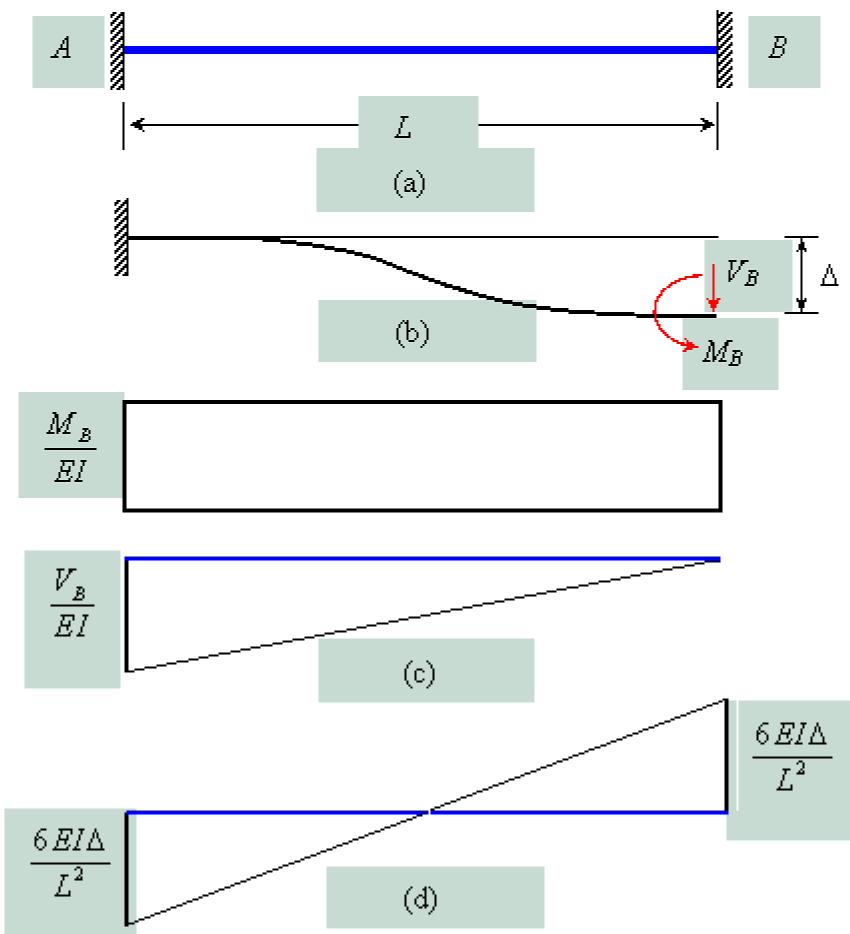


Figure 5.6(a)-(d)

Applying first moment area theorem between A and B

$$\Delta\theta_{AB} = L \times \frac{M_B}{EI} - \frac{1}{2} \times \frac{V_B L}{EI} \times L = 0$$

$$M_B = \frac{V_B L}{2}$$

Applying second moment area theorem between point A and B

$$t_{BA} = \frac{M_B}{EI} \times L \times \frac{L}{2} - \frac{1}{2} \times \frac{V_B L}{EI} \times L \times \frac{2L}{3}$$

$$-\Delta = \frac{M_B L^2}{2EI} - \frac{V_B L^3}{3EI}$$

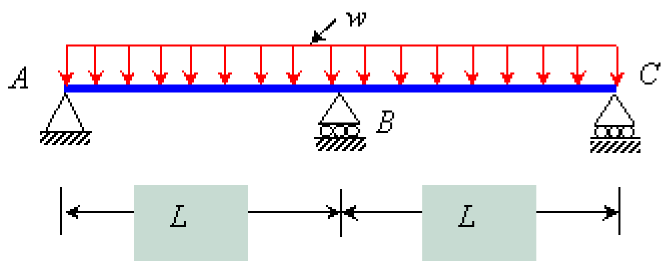
Solving eqs. (i) and (ii)

$$M_B = \frac{6EI\Delta}{L^2} \quad \text{and} \quad V_B = \frac{12EI\Delta}{L^3}$$

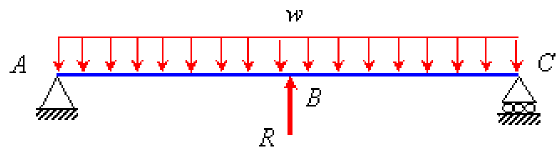
By equilibrium conditions, the reactions at support A are

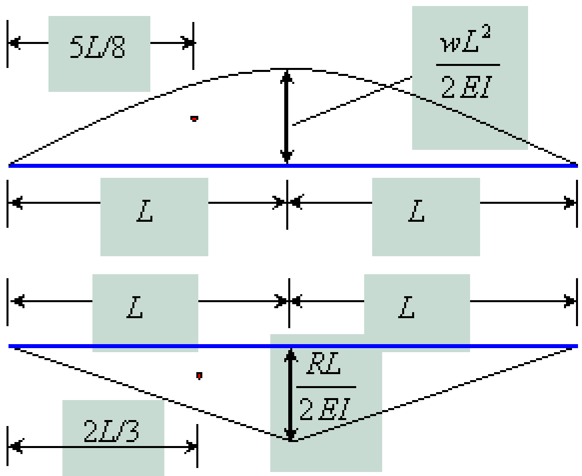
$M_A = \dfrac{6EI\Delta}{L^2}$ And $V_A = \dfrac{12EI\Delta}{L^3}$

Determine the support reactions of the continuous beam as shown in Figure 7(a)



The static indeterminacy of the beam is = 3-2 =1. Let the vertical reaction at B be the unknown R as shown in Figure 7(b). The M/EI diagrams of the beam are shown in Figure 7(c)





Because of symmetry of two spans the slope at B , $\theta_B = 0$. As a result

$$t_{AB} = 0$$

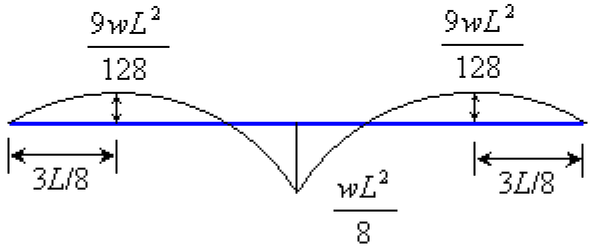
$$t_{AB} = \left(\frac{2}{3} \times \frac{wL^2}{2EI} \times L\right)\frac{5L}{8} - \left(\frac{1}{2} \times \frac{RL}{2EI} \times L\right)\frac{2L}{3} = 0$$

$$R = \frac{5wL}{4}$$

The vertical reaction at A and C are

$$V_A = V_C = wL - \frac{1}{2} \times \frac{5wL}{4} \quad = \frac{3}{8} wL$$

The bending moment diagram of the beam is shown in Figure 7(d)



Derivation of three moment equation for analysis of continuous beams.

Three Moment Equation

The continuous beams are very common in the structural design and it is necessary to develop simplified force method known as three moment equation for their analysis. This equation is a relationship that exists between the moments at three points in continuous beam. The points are considered as three supports of the indeterminate beams. Consider three points on the beam marked as 1, 2 and 3 as shown in Figure. Let the bending moment at these points are M_1, M_2, M_3 and the corresponding vertical displacement of these points are $\Delta_1, \Delta_2, \Delta_3$ respectively. Let L_1, L_2 be the distance between points 1 – 2 and 2 – 3, respectively.

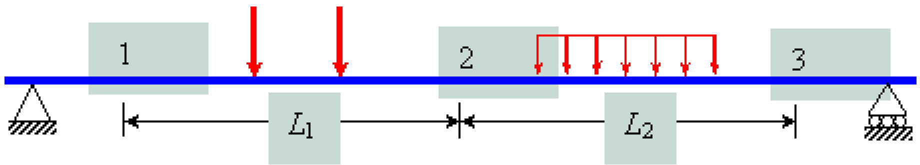


Figure 5.25(a)

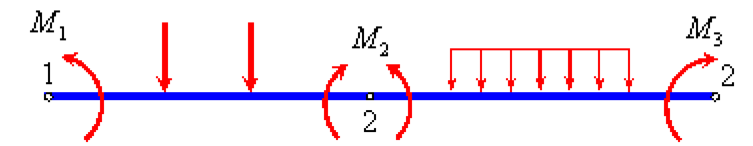


Figure 5.25(b)

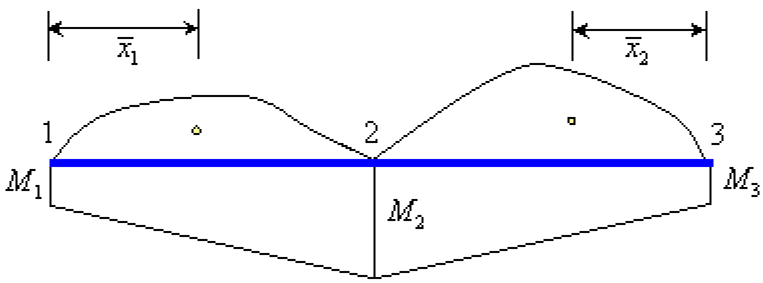


Figure 5.25(c)

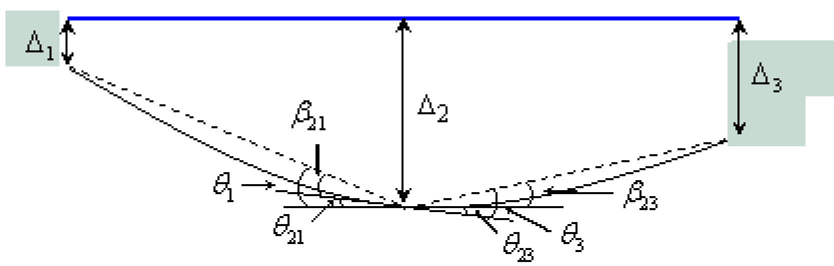


Figure 5.25(d)

The continuity of deflected shape of the beam at point 2 gives $\theta_{21} = \theta_{23}$

From the Figure 5.25(d)

$$\theta_{21} = \theta_1 - \beta_{21}, \quad \theta_{23} = \theta_3 - \beta_{23}$$

$$\theta_1 = \frac{\Delta_1 - \Delta_2}{L_1}, \quad \theta_3 = \frac{\Delta_3 - \Delta_2}{L_2}$$

Using the bending moment diagrams shown in Figure 5.25(c) and the second moment area theorem

$$\theta_{21} = \frac{1}{L_1} \times \frac{1}{EI_1}\left(\frac{M_1 L_1^2}{6} + \frac{M_2 L_1^2}{3} + A_1\bar{x}_1\right)$$ And $$\theta_{23} = \frac{1}{L_2} \times \frac{1}{EI_2}\left(\frac{M_3 L_1^2}{6} + \frac{M_2 L_1^2}{3} + A_2\bar{x}_2\right)$$

where A_1 and A_2 are the areas of the bending moment diagram of span 1-2 and 2-3, respectively considering the applied loading acting as simply supported beams. Upon substituting the values and rearranging gives

$$M_1\left(\frac{L_1}{I_1}\right) + 2M_2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_3\left(\frac{L_2}{I_2}\right) = -\frac{6A_1\bar{x}_1}{I_1L_1} - \frac{6A_2\bar{x}_2}{I_2L_2} + 6E\left[\frac{(\Delta_2 - \Delta_1)}{L_1} + \frac{(\Delta_2 - \Delta_3)}{L_2}\right]$$

The above is known as **three moment equation or Claypron theorem.** .

Sign Conventions

M_1, M_2, M_3 are positive for sagging moment and negative for hogging moment. Similarly, areas $A_1, A_2, A_3$are positive if it is sagging moment and negative for hogging moment. The displacements $\Delta_1, \Delta_2, \Delta_3$ are positive if measured downward from the reference axis.

Analyze the continuous beam shown in Figure 5.26(a) by the three moment equation. Draw the shear force and bending moment diagram

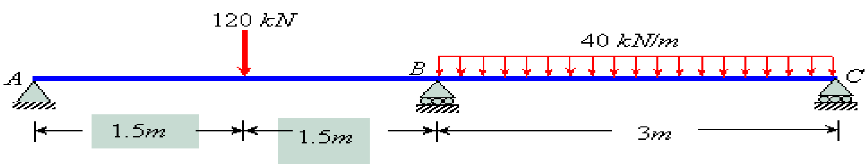


Figure 5.26(a)

The simply supported bending moment diagram on AB and AC are shown in Fig 5.26 (b). Since supports A and C are simply supported.

$M_A = M_C = 0$

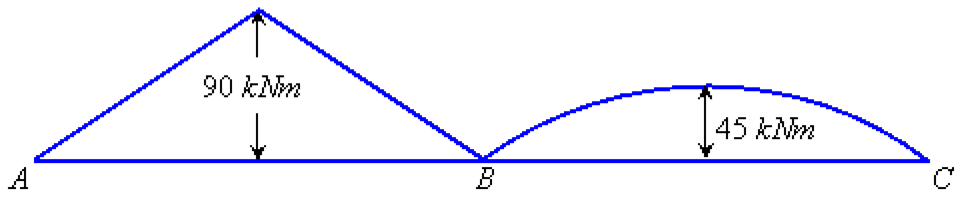


Figure 5.26(b)

Applying the three moment equation to span AB and BC $\Delta_1 = \Delta_2 = \Delta_3 = 0$

$$M_A\left(\frac{3}{I}\right) + 2M_B\left(\frac{3}{I} + \frac{3}{I}\right) + M_C\left(\frac{3}{I}\right) = -\frac{6 \times 1/2 \times 90 \times 3 \times 1.5}{3 \times I} - \frac{6 \times 2/3 \times 45 \times 3 \times 1.5}{3 \times I}$$

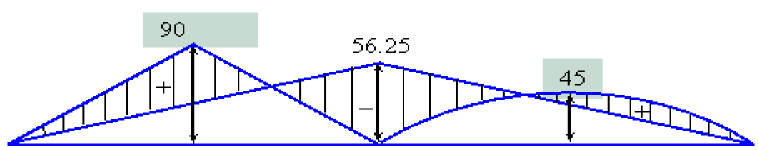
$$M_B = -56.25KN$$

The reactions at support A , B and C are given as

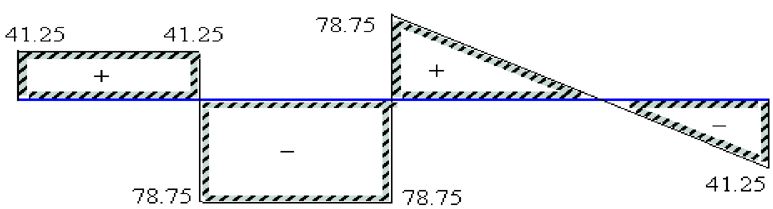
$$V_A = \frac{120 \times 1.5 - 56.25}{3}, \quad V_C = \frac{40 \times 3 \times 1.5 - 56.25}{3},$$

$$V_B = 120 + 40 * 3 - 41.25 - 41.25 = 157.5KN$$

The bending moment and shear force diagram are shown in Figures 5.26(c) and (d), respectively



(c)



(d)

Deflection by Strain Energy Method

The concepts of strain, strain-displacement relationships are very useful in computing energy-related quantities such as work and strain energy. These can then be used in the computation of deflections. In the special case, when the structure is linear elastic and the deformations are caused by external forces only, (the complementary energy U * is equal to the strain energy U) the displacement of structure in the direction of force is expressed by

$$\Delta_j = \frac{\partial U}{\partial P_j}$$

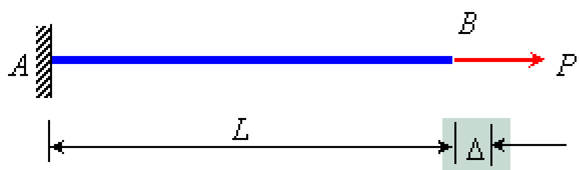
This equation is known as Castigliano's theorem. It must be remembered that its use is limited to the calculation of displacement in linear elastic structures caused by applied loads. The use of this theorem is equivalent to the virtual work transformation by the unit-load theorem.

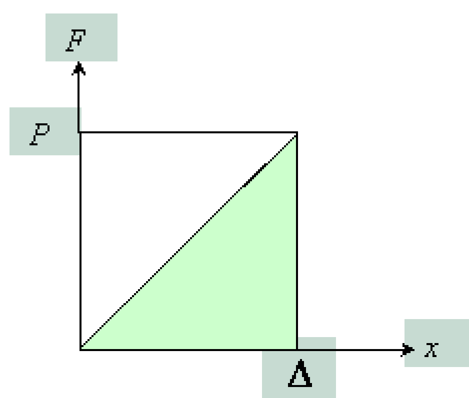
Calculation of Strain Energy

When external loads are applied on an elastic body they deform. The work done is transformed into elastic strain energy U that is stored in the body. We will develop expressions for the strain energy for different types of loads.

Axial Force : Consider a member of length L and axial rigidity AE subjected to an axial force P applied gradually as shown in the Figure. The strain energy stored in the member will be equal to the external work done by the axial force i.e

$$U = \frac{1}{2} \times P \times \Delta = \frac{1}{2} \times P \times \frac{PL}{AE} = \frac{P^2 L}{2AE}$$



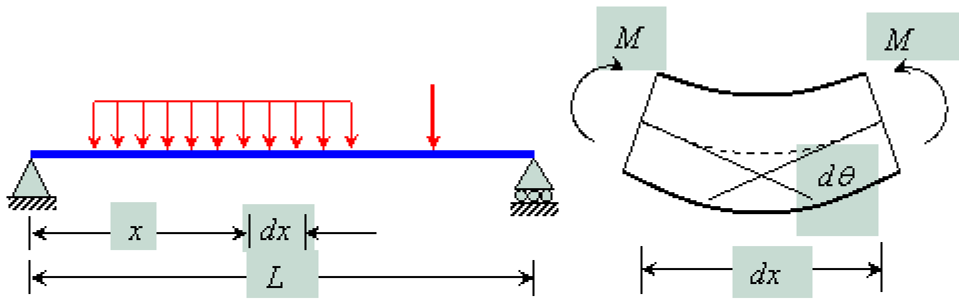


Bending Moment: Consider a beam of length L and flexural rigidity EI subjected to a general loading as shown in Figure. Consider a small differential element of length, dx . The energy stored in the small element is given by

$$dU = \frac{1}{2} \times M \times d\theta = \frac{1}{2} \times M \times \frac{M}{EI} dx = \frac{M^2}{2EI} dx$$

The total strain energy in the entire beam will be

$$U = \int_0^L \frac{M^2}{2EI} dx$$



Shear Force: The strain energy stored in the member due to shearing force is expressed by

$$U = \int_0^L \frac{V^2}{2GA_s} dx$$

where *V* is the shearing force; and GA_s is the shearing rigidity of the member.

Twisting Moment: The strain energy stored in the member due to twisting moment is expressed by

$$U = \int_0^L \frac{T^2}{2GJ} dx$$

where *T* is the twisting moment; and *GJ* is the torsional rigidity of the member.

Example 4.18 Find the horizontal deflection at joint *C* of the pin-jointed frame as shown in Figure 4.26(a). *AE* is constant for all members.

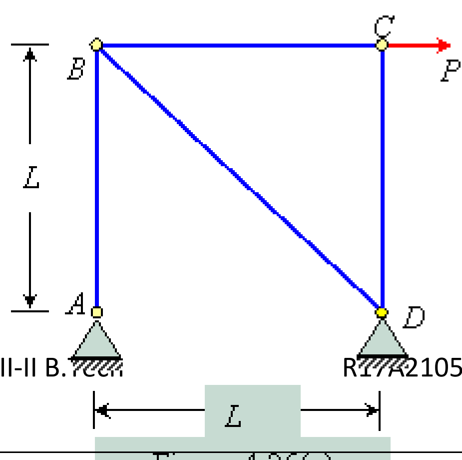


Figure 4.26(a)

Solution: The force in various members of the frame is shown in Figure 4.26(b). Calculation of strain energy of the frame is shown in Table 4.4.

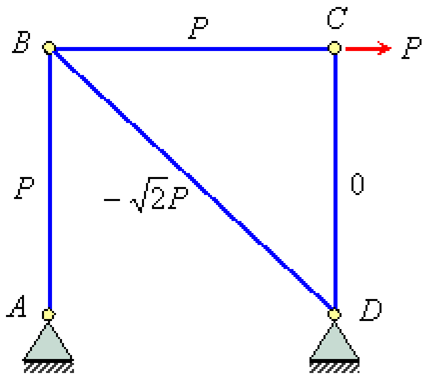


Figure 4.26(b). Forces due to applied load

Σ $(\sqrt{2}+1)P^2L/AE$

Horizontal displacement of joint *C*

$$\Delta_{CH} = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left(\frac{(\sqrt{2}+1)P^2L}{AE}\right)$$

$$= \frac{2(\sqrt{2}+1)PL}{AE}(\rightarrow)$$

Example 4.19 A bar of uniform cross-section is bent into a quadrant of circle of radius *R* . One end of the bent is fixed and other is free. At the free end it carries a vertical load *W* . Determine the vertical and horizontal deflection at *A*.

Solution:

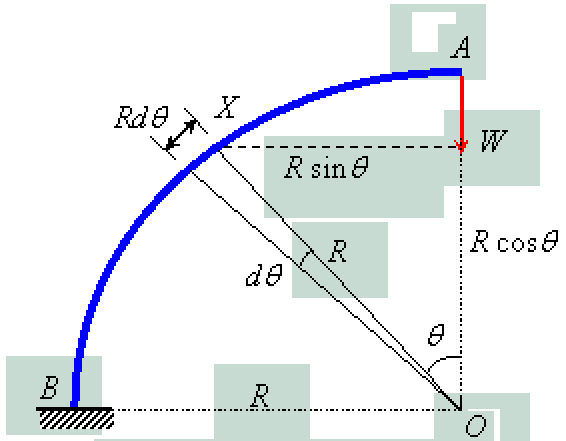


Figure 4.27(a)

Vertical displacement of A : The vertical displacement of *A* is given by

$$\Delta_{AV} = \frac{\partial U}{\partial W}$$

For evaluation of the total strain energy in the system, consider a small element $Rd\theta$ as shown in the Figure. The bending moment at this element,

$$M_\theta = -W R \sin\theta$$

Thus,

$$\Delta_{AV} = \frac{\partial}{\partial W}\left(\int_0^{\pi/2} \frac{M_\theta^2 R d\theta}{2EI}\right)$$

$$= \frac{1}{EI}\int_0^{\pi/2} M_\theta \left(\frac{\partial M_\theta}{\partial W}\right) R d\theta$$

$$= \frac{1}{EI}\int_0^{\pi/2} WR^3 \sin^2\theta \, d\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{\pi W R^3}{4EI}$$

Since there is no horizontal force acting at point *A* , apply a horizontal force, *F* at *A* as shown in Figure 4.27(b). From the Castigliano's theorem, the horizontal displacement of *A* due to applied external load *W* is given by

$$\Delta_{AH} = \left.\frac{\partial U}{\partial F}\right|_{F=0}$$

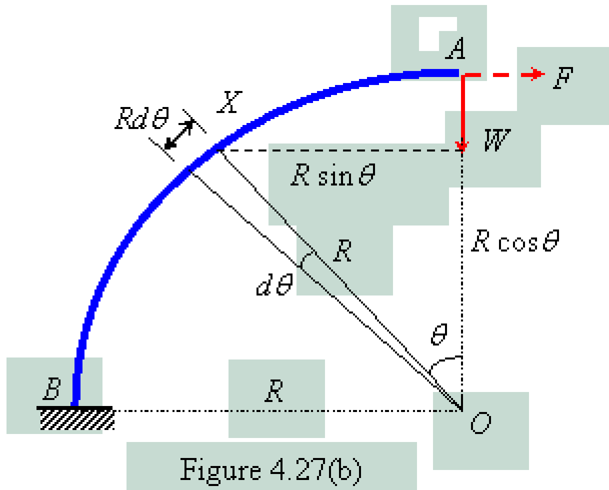


Figure 4.27(b)

The bending moment at the small element

$$M_\theta = -WR\sin\theta - FR(1-\cos\theta)$$

Thus, the horizontal displacement of *A*

$$\Delta_{AH} = \left.\frac{\partial}{\partial F}\left(\int_0^{\pi/2} \frac{M_\theta^2 R d\theta}{2EI}\right)\right|_{F=0}$$

$$= \frac{1}{EI}\int_0^{\pi/2} M_\theta \left.\left(\frac{\partial M_\theta}{\partial F}\right) R\, d\theta\right|_{F=0}$$

$$= \frac{1}{EI}\int_0^{\pi/2} \left(-WR\sin\theta - FR(1-\cos\theta)\right)\left.\left(\frac{\partial\left(WR\sin\theta - FR(1-\cos\theta)\right)}{\partial F}\right) R\, d\theta\right|_{F=0}$$

$$= \frac{1}{EI}\int_0^{\pi/2} \left(-WR\sin\theta\right)\left(-R(1-\cos\theta)\right) R\, d\theta$$

$$= \frac{1}{EI}\int_0^{\pi/2} \left[-WR^3\left(-\sin\theta + \frac{\sin 2\theta}{2}\right)\right] d\theta$$

$$= -\frac{WR^3}{2EI}$$

Example 4.20 Determine the deflection of the end *A* of the beam as shown in Figure 4.28. The flexibility of the spring is

$f = L^3 / EI$

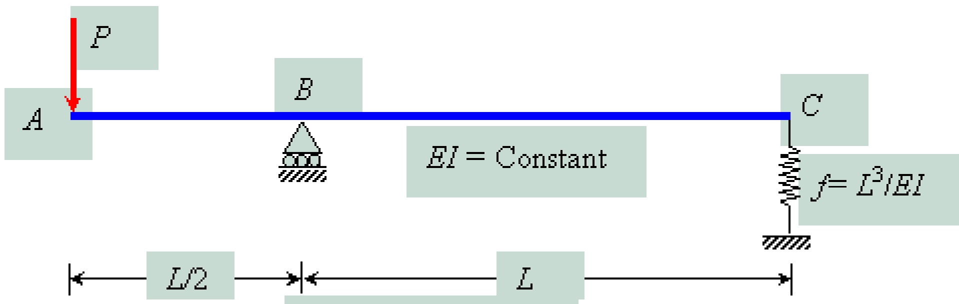


Figure 4.28

Solution: Reactions at support *B* and *C* are

$R_B = 1.5P$ (upward) and $R_C = 0.5P$ (downward)

Force in the spring = Reaction, $R_C = 0.5P$

Deflection under the load is given by

$$\Delta_A = \frac{\partial U}{\partial P}$$

where

$U = U_{AB} + U_{BC} + U_S$ is the total strain energy stored in the system

U_{AB} is the energy stored in the member *AB*

U_{BC} is the energy stored in the member *BC*

and

U_S is the strain energy in the spring.

$$U_S = \frac{1}{2} f \left(\frac{P}{2}\right)^2 = \frac{P^2L^3}{8EI}$$

Consi

der member *AB* : (*x* measured from *A*)

$$M_x = -Px$$

$$U_{AB} = \int_0^{L/2} \frac{M_x^2 dx}{2EI} = \int_0^{L/2} \frac{P^2x^2dx}{2EI} = \frac{P^2L^3}{48EI}$$

Consider member *BC* : (*x* measured from *C*)

$$M_x = -\frac{P}{2}x$$

$$U_{BC} = \int_0^{L} \frac{M_x^2 dx}{2EI} = \int_0^{L} \frac{P^2x^2dx}{8EI} = \frac{P^2L^3}{24EI}$$

Thus

$$U = U_{AB} + U_{BC} + U_S$$

$$= \frac{P^2L^3}{8EI} + \frac{P^2L^3}{48EI} + \frac{P^2L^3}{24EI} = \frac{3P^2L^3}{16EI}$$

The deflection of point *A*,

$$\Delta_A = \frac{\partial U}{\partial P} = \frac{3PL^3}{8EI}$$